INFINITIES

A MEASURE OF OUR IGNORANCE

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"INFINITIES AND COSMOLOGY " WORKSHOP
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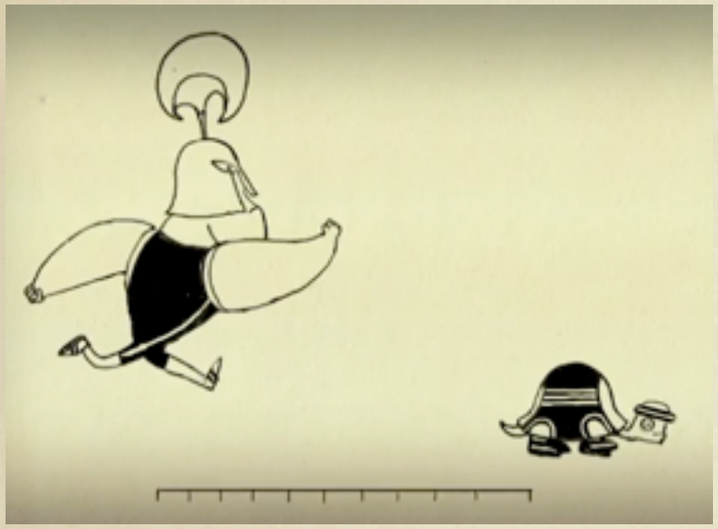
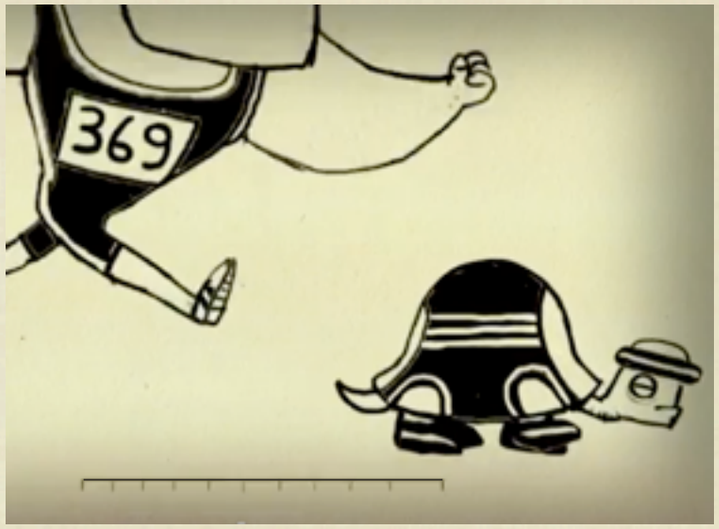
Becoming and Discreteness

Zeno's paradox

Zeno's paradox





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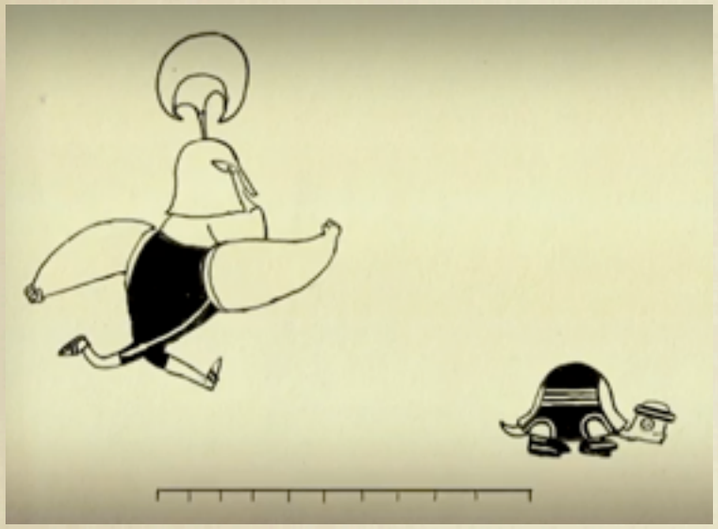
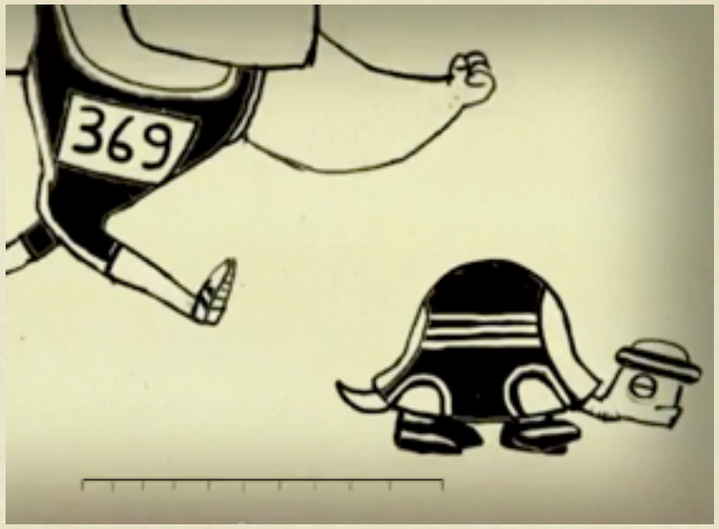
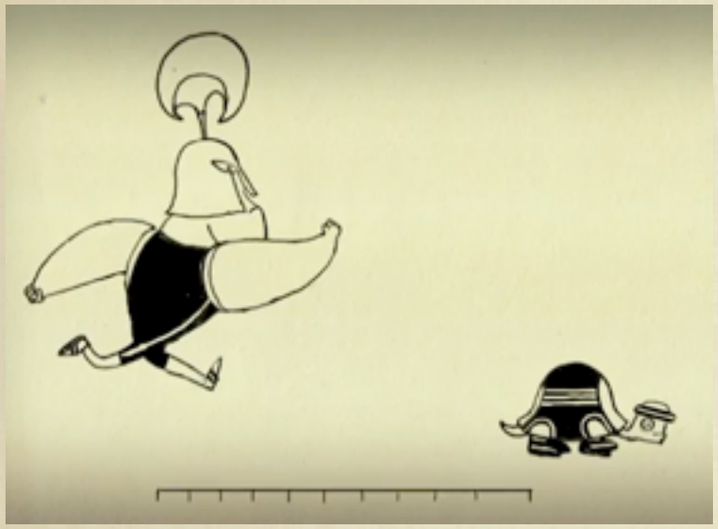
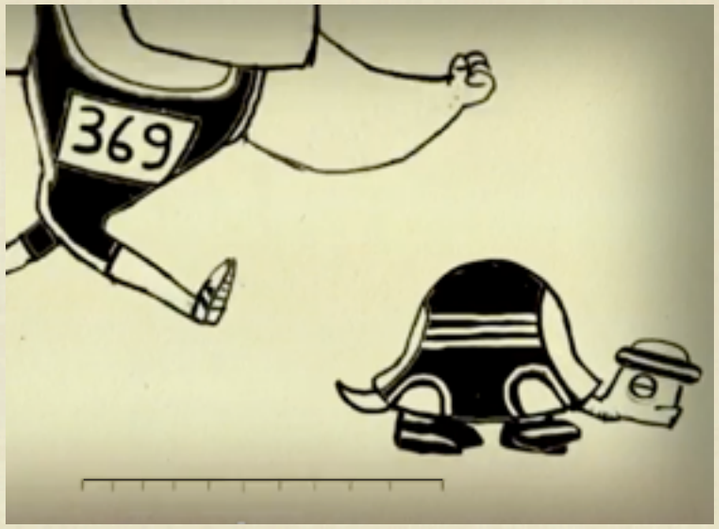




$$\sum_{n=1}^{\infty} (1/2)^n = 1$$

Zeno's paradox





$$\sum_{n=1}^{\infty} (1/2)^n = 1$$

"Zeno's arguments, in some form, have afforded grounds for almost all theories of space and time and infinity which have been constructed from his time to our own." Bertrand Russell

Becoming and Discreteness

being

vs

becoming

continuous

vs

discrete

Becoming and Discreteness

being

vs

becoming

continuous

vs

discrete

Becoming and Discreteness

being

vs

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continuous

vs

discrete

Relational and Discrete *(Rovelli's program)*

- **RELATIONAL INTERPRETATION OF QUANTUM MECHANICS**
  ontological meaning is shifted away from the wave function
  in favor of the measurement process

- **THERE IS NO ABSOLUTE TIME**
  but many relational times governed by interactions

- **QUANTA OF SPACETIME**
  prediction of Loop Quantum Gravity
  fundamental scale: the Planck length

Constants of Nature

Constants exists in nature

- **Special Relativity**
  maximal local physical velocity: the speed of light c

- **Quantum Mechanics**
  minimal action $\hbar$ in all physical interactions
  $\equiv$ a finite region of phase space contains only
  a finite number of distinguishable (orthogonal) quantum states
  $\equiv$ minimal amount of information in the state of a system

- **Quantum Gravity**
  minimal length: the Planck length ℓ_P

The Planck length

- Quantum Mechanics

  Heisenberg Uncertainty

  Sharp localization requires large energy.

  $$\Delta x > \hbar / \Delta p$$

  $$E \sim cp$$

- General Relativity

  Black-Hole Horizon

  The horizon prevent a sharper localization.

  $$M \sim E/c^2$$

  $$R \sim GM/c^2$$

  $$\Delta x \geq R$$

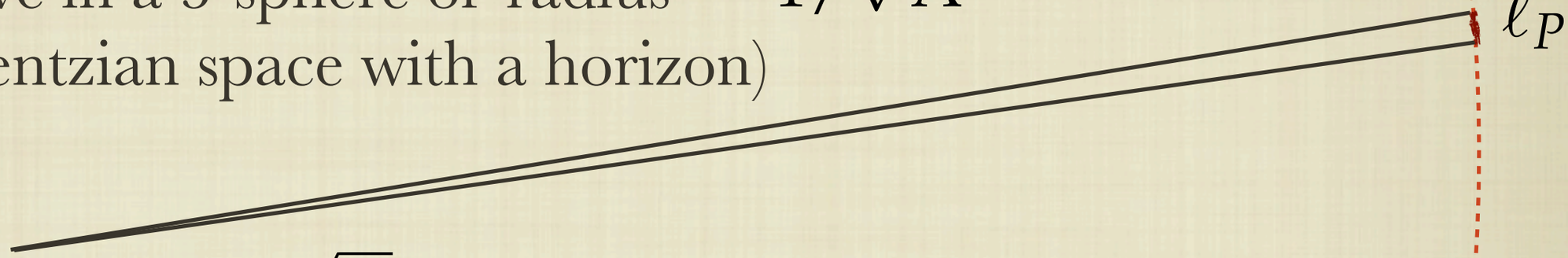
- Quantum Gravity

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35}\, m$$

The Cosmological Constant

- It is part of GR formulation
- We measure it!

- If we live in a 3-sphere of radius $\sim 1/\sqrt{\Lambda}$
(or Lorentzian space with a horizon)



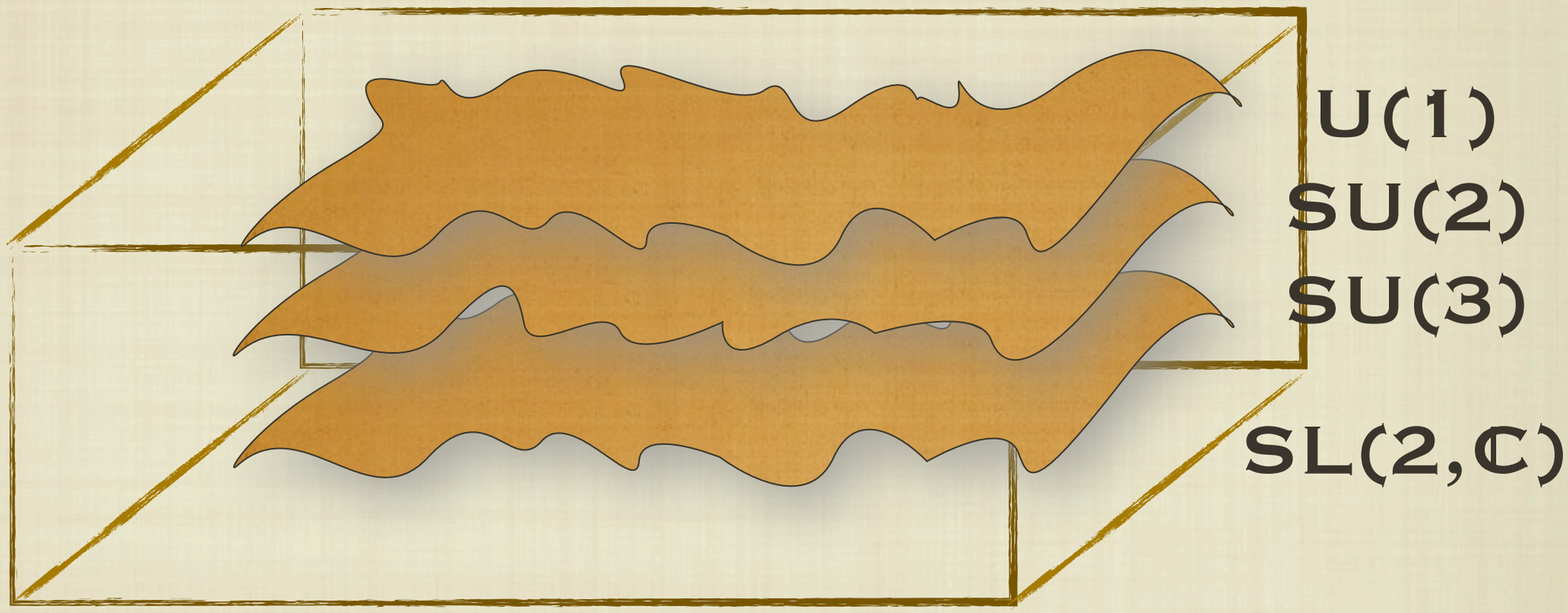
- Spherical harmonics with $j \leq j_{max} = 4\pi/\phi^2_{min} \sim \dfrac{4\pi}{\ell_P^2\,\Lambda}$

- Quantization of geometry: angles should be quantized!
Λ: minimal resolution of small angles in the sky

General Covariance

The Gravitational Field

- **General Relativity:** background independence!



- fields <---> Gauge Symmetries

The Gravitational Field

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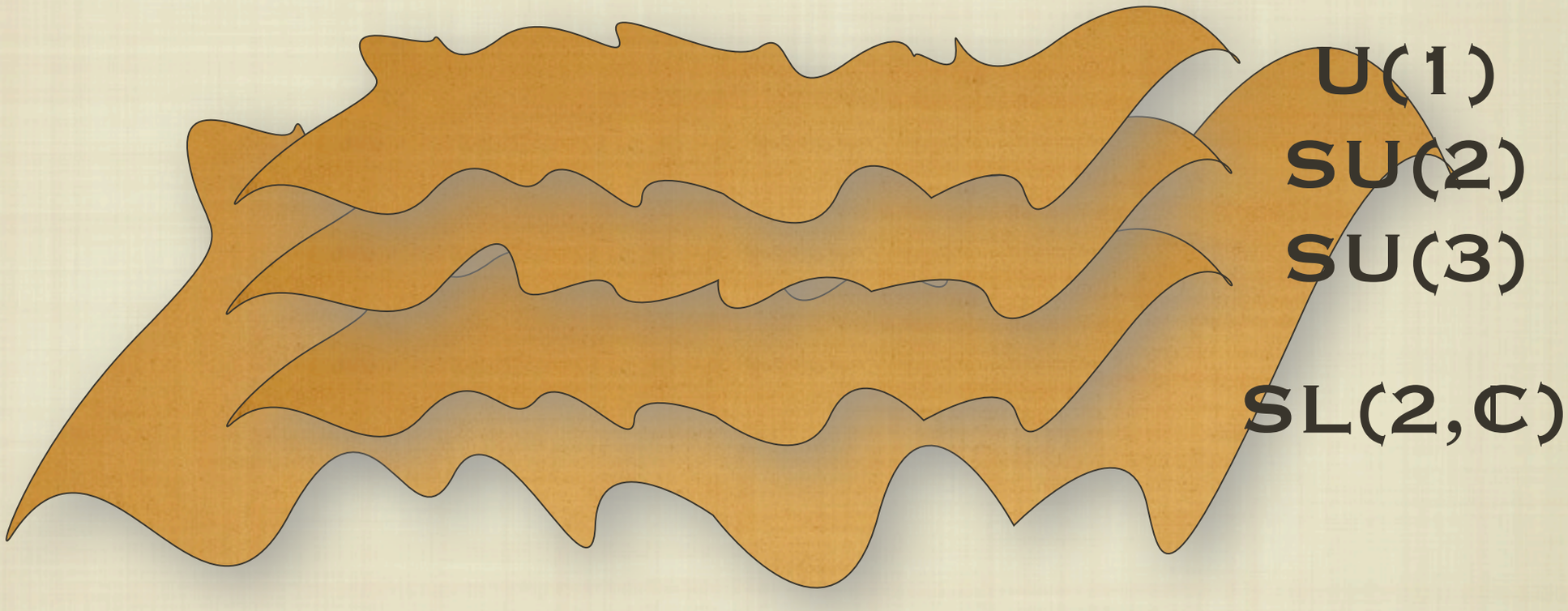

- fields <---> Gauge Symmetries
- Gravity as an interacting gauge field

Classical theory

Tetrads $\quad g_{ab} \to e^i_a \qquad g_{ab} = e^i_a \; e^i_b \qquad e = e_a dx^a \in \mathbb{R}^{(1,3)}$

Spin connection $\quad \omega = \omega_a dx^a \in sl(2,\mathbb{C}) \qquad \omega(e): \; de + \omega \wedge e = 0$

Holst action $\quad S[e,\omega] = \int e \wedge e \wedge F^*[\omega] + \frac{1}{\gamma} \int e \wedge e \wedge F[\omega]$

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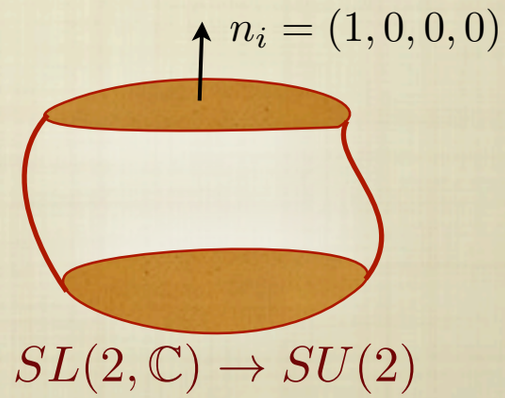
- **Gravity as an interacting gauge field**
  but only in the tetrad formulation we can couple fermions and Yang-Mills fields!

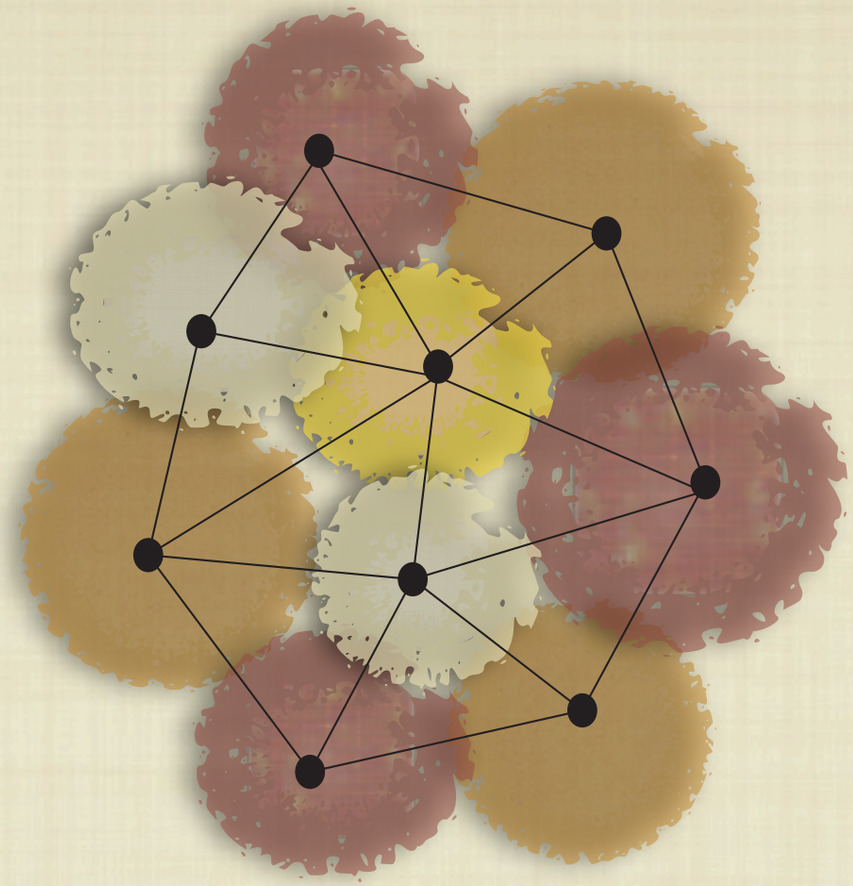
Lorentz Invariance & Time Gauge

- General Relativity has a local Lorentz invariance
- In each point of spacetime, we have a Lorentz invariant tetrad
- Time is a pure Gauge: we can fix it
- We are left with the rotational part of the Lorentz transformations
- On the boundary $n_i = e^a_i n_a \qquad n_i e^i = 0$
- Linear simplicity constraint $\vec{K} + \gamma \vec{L} = 0$

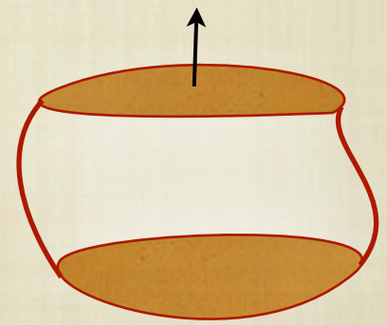


Quanta of Spacetime

Loop Quantum Gravity

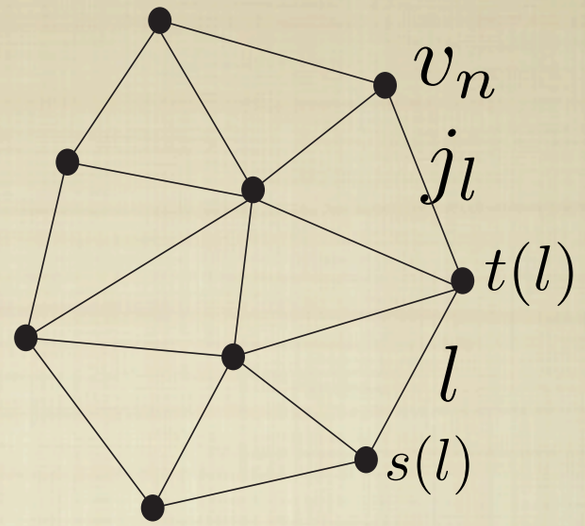


- It is a theory about quanta of spacetime
- Each quantum is Lorentz invariant
- The states are boundary states at fixed time
- The physical phase space is spanned by SU(2) group variables



$SL(2,\mathbb{C}) \to SU(2)$

Hilbert Space



- Abstract graphs: $\Gamma=\{N,L\}$
- Group variables: $\begin{cases} h_l \in SU(2) \\ \vec{L}_l \in su(2) \end{cases}$
- Graph Hilbert space: $\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N]$
- The space $\mathcal{H}_\Gamma$ admits a basis $|\Gamma, j_\ell, v_n\rangle$
- Gauge invariant operator $G_{ll'} = \vec{L}_l \cdot \vec{L}_{l'}$ with $\sum_{l\in n} G_{ll'} = 0$

  Penrose's spin-geometry theorem (1971), and Minkowski theorem (1897)

- Geometry is quantized:
  - eigenvalues are discrete
  - the operators do not commute
  - quantum superposition

Quantum states of space
rather than states on space

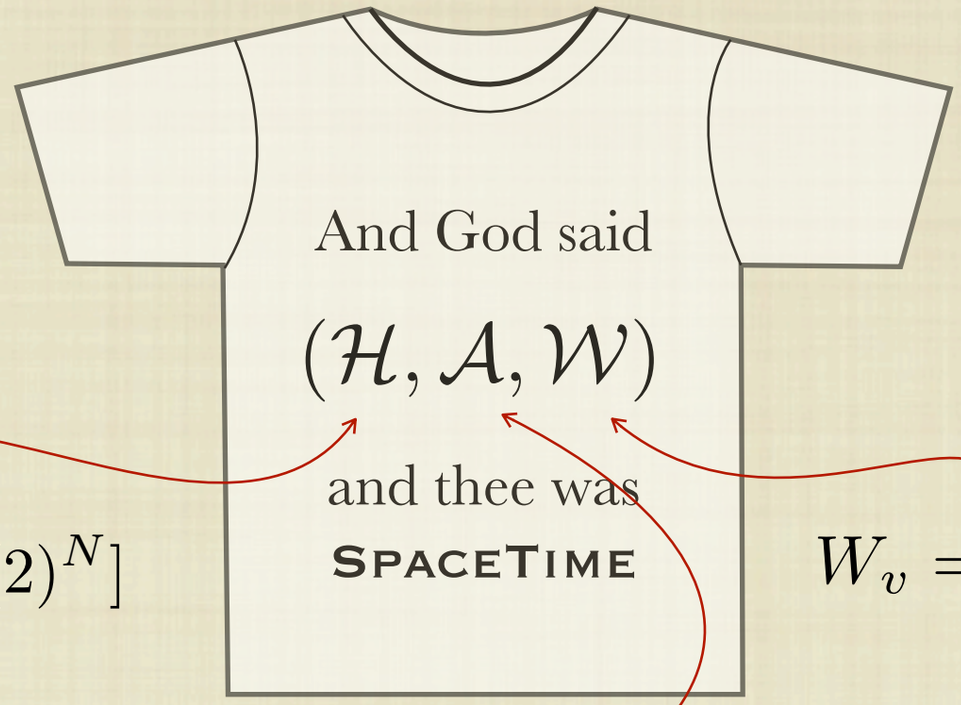
Loops and Discrete Geometry

- h_l "Holonomy of the Ashtekar-Barbero connection along the link"
- $\vec{L}_l = \{L^i_l\}, i = 1, 2, 3$ SU(2) generators

  gravitational field operator (tetrad) $L^i\psi(h) \equiv \left.\frac{d}{dt}\psi(he^{t\tau_i})\right|_{t=0}$

- Composite operators:
  - Area: $A_\Sigma = \sum_{l\in\Sigma}\sqrt{L^i_l L^i_l}.$
  - Volume: $V_R = \sum_{n\in R} V_n, \qquad V_n^2 = \frac{2}{9}\,|\epsilon_{ijk}L^i_l L^j_{l'} L^k_{l''}|.$
  - Angle: $L^i_l L^i_{l'}.$

Definition of the Theory



Hilbert Space:

$$\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N]$$

Transition Amplitude:

$$W_v = (P_{SL(2,\mathbb{C})} \circ Y_\gamma\ \psi_v)(\mathbb{1})$$

Operators' Algebra:

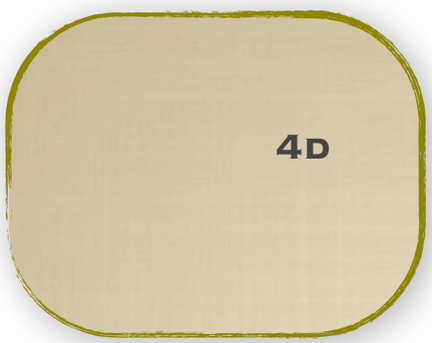
$$[L_a^i, L_b^j] = i\delta_{ab} l^2 \varepsilon^{ij}{}_k\ L_a^k$$

Covariant Transition Amplitudes

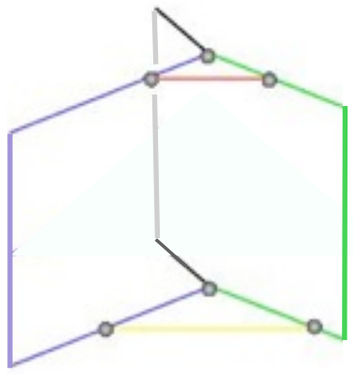
Spinfoam Amplitudes [Engle-Pereira-Livine-Rovelli, Freidel-Krasnov '08]

Probability amplitude $P(\psi) = |\langle W|\psi\rangle|^2$

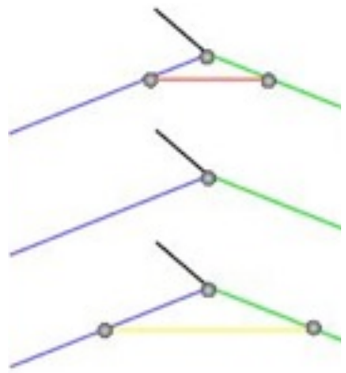
Amplitude associated to a state ψ of a boundary of a 4d region



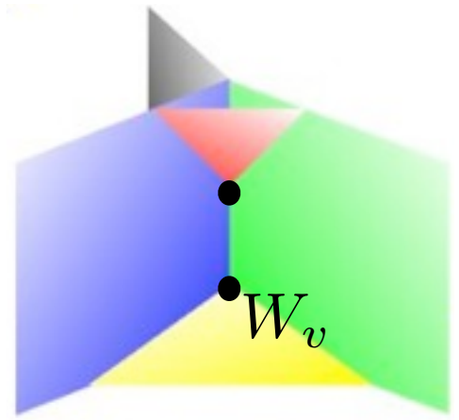
3D boundary



boundary graph



a spin network history



σ : spinfoam

- Superposition principle $\langle W|\psi\rangle = \sum_\sigma W(\sigma)$
- Locality: vertex amplitude $W(\sigma) \sim \prod_v W_v.$
- Lorentz invariance $W_v = (P_{SL(2,\mathbb{C})} \circ Y_\gamma\ \psi_v)(\mathbb{1})$

LARGE DISTANCE LIMIT

$$W_{\mathcal{C}}(h_l) = \int_{SU(2)} dh_{vf} \prod_f \delta(h_f) \prod_v A(h_{vf})$$

Theorem :
[Barrett, Pereira, Hellmann, Gomes, Dowdall, Fairbairn 2010]

$$A(j_f, i_e) \underset{j\gg 1}{\sim} e^{iS_{\text{Regge}}} + e^{-iS_{\text{Regge}}}$$

[Freidel Conrady 2008, Bianchi, Satz 2006, Magliaro Perini, 2011]

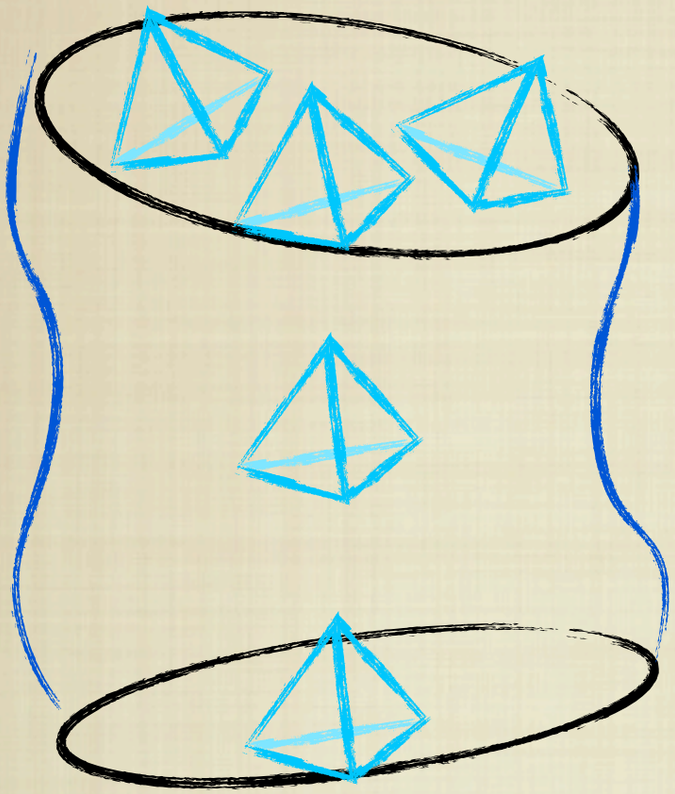
$$W_{\mathcal{C}} \xrightarrow[j\gg 1]{} e^{iS_\Delta} \qquad Z_{\mathcal{C}} \xrightarrow[C\to\infty]{} \int Dg \; e^{iS[g]}$$

Theorem :
[Han 2012]

$$A^q(j_f, i_e) \underset{j\gg 1, q\sim 1}{\sim} e^{iS^\Lambda_{\text{Regge}}} + e^{-iS^\Lambda_{\text{Regge}}} \qquad q = e^{\Lambda\hbar G}$$

Transition Amplitudes in Cosmology

Bianchi, Rovelli, FV '10



q'_{ij}

q_{ij}

$$W(q'_{ij}, q_{ij}) \sim \int_{\partial g = q', q} Dq \; e^{i\,S}$$

Two-complex: $\mathcal{C} = \{V, E, F\}$

- expansion: discretization of spacetime

  graph: truncation of the # degrees of freedom

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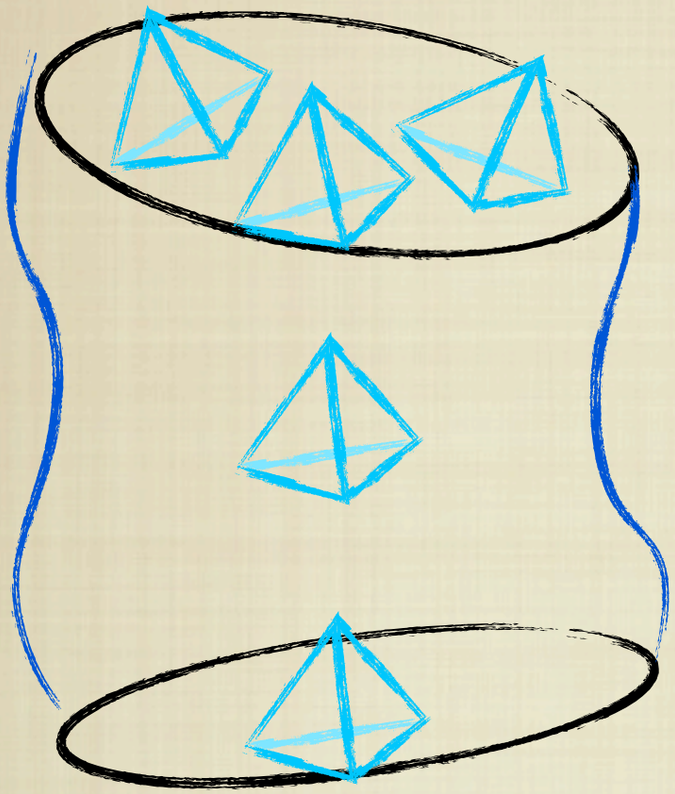

q'_{ij}

the universe at "time" t'

q_{ij}

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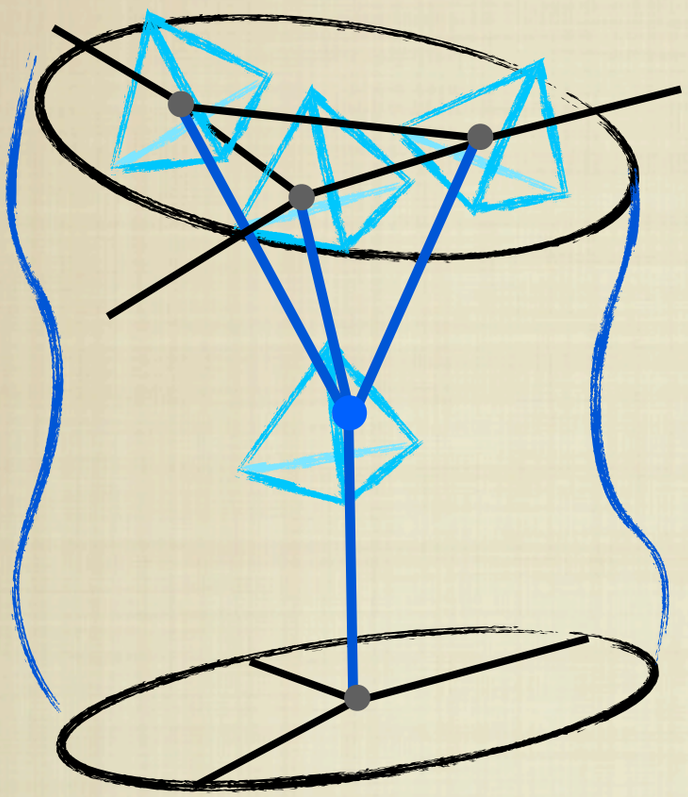

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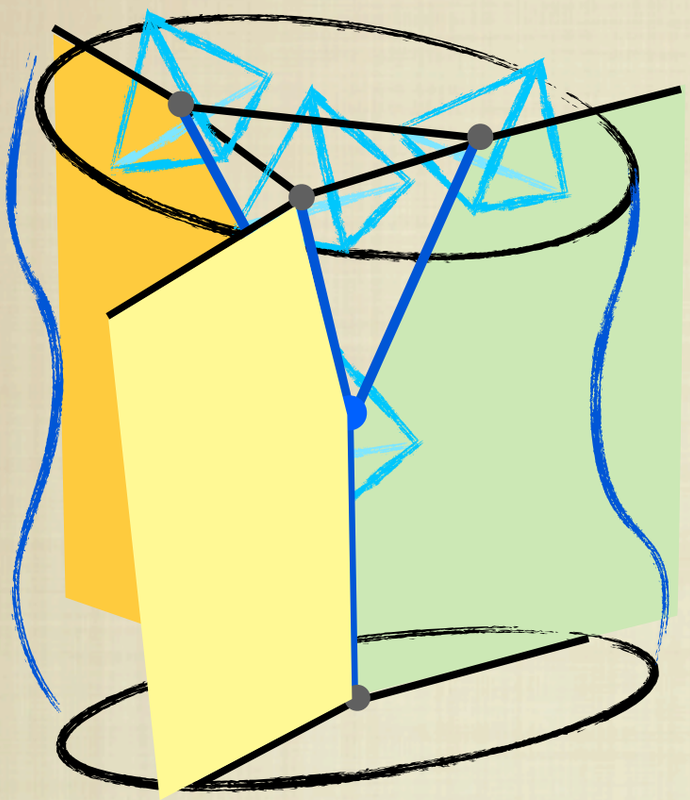

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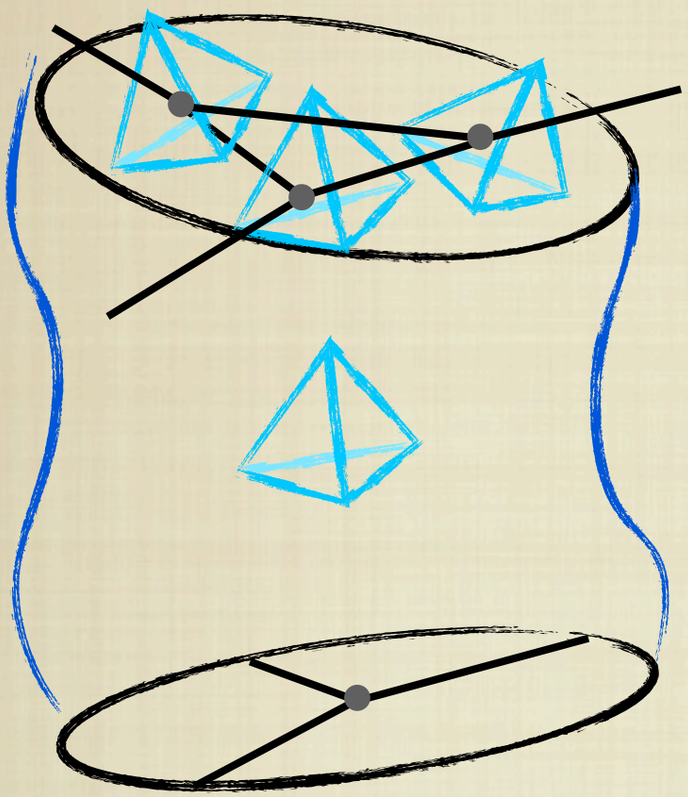

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Covariant Loop Gravity (Spinfoam)

- The boundary states represent classical geometries.
  Canonical LQG 1990, Penrose spin-geometry theorem 1971
- Boundary geometry operators have discrete spectra.
  Canonical LQG main results, 1990
- The classical limit of the vertex amplitude converges to the Regge Hamilton function (with cosmological constant).
  Barrett et al, Conrady-Freidel, Bianchi-Perini-Magliaro, Engle, Han..., 2009-2012
- Amplitudes are locally Lorentz covariant.
  The short-scale discrete geometry does not break Lorentz invariance.
- Theory extended to fermions and Yang Mills fields.
  Bianchi, Han, Magliaro, Perini, Rovelli, Wieland 2010
- The amplitudes with positive Λ are UV and IR finite.
  Han, Fairbairn, Moesburger, 2011

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Perturbative vs Non-Perturbative

- Early perturbative quantum gravity: **non-renormalizability**
  - Local: observables at arbitrarily small regions in a continuous manifold
  - **Infinite** renormalization group
  - Cut-off: it is a mathematical trick
- Perturbations methods are some kind of approximation.
- Infinities: we perturb around points that are not really good.
- Non-perturbative approach: presence of a fundamental scale!

Quantum Groups

- Planck length + horizon = minimal angular resolution
- Mathematically a *fuzzy spheres:* spherical harmonics with j_{max}
- A maximum angular momentum characterizes the representations of the quantum group $SU(2)_q$ (Majid'88)

$$q = e^{i2\pi/k} \quad \text{with k}\sim 2\, j_{max}$$

- The local rotational symmetry is better described by than by SU(2), with $q = e^{i\Lambda l_P^2}$
- Physically: non-commutativity, fuzziness of any angular function, impossibility of resolving small dihedral angles. (Connes'94)
- Loop gravity: ϕ is an operator with a discrete spectrum. (Major'99)
- Best angular resolution: $\phi_{min} = \sqrt{2/j_{max}}$ with $j_{max} \sim \frac{1}{l_P^2\,\Lambda}$

Holy Infinities

Numbers and the Sacred

- Sacred is everything that we feel such as not belonging to human world (Eliade '57). It could be a space, it could be a time, it could be everything that goes beyond our human experience.

- In this world, we size things with respect to us: we say heavens are up in the sky because we are confined down by gravity, we measure land in terms of our foot, we count as much as needed by our daily experience.

- Modeling, sizing and counting are sophisticated tools that we have developed but are not given a priori.
  Is there something that we can not model/size/count?

- We call in this way what cannot be said.
  We call infinite what is beyond our human experience.
  But if this is a condition of our human nature, it has not to be something that has necessarily to exist in Nature.

- Infinity can exist as a mathematical object, a powerful tool for our calculation. But physics is not just mathematics. It is a discourse about nature trough the mathematical language, where the ultimate goal is to associate a number to a physical system, and from this number a meaning.

- An infinity has the meaning of *"beyond our present knowledge"*. But making science and making physics is to constantly push forward the boundary of our knowledge, in a process that cannot have an end.

``There are some,
king Gelon, who think that
the number of the sand is
infinite in multitude"

Archimedes
“Sand Reckoner"

All wisdom comes from Yahweh
and with him it remains forever.
The sand of the seashore, and the drops of rain,
and the days of eternity: who can number these?
Heaven's height, earth's breadth,
the depths of the abyss: who can explore these?
Before all other things wisdom was created;
and prudent understanding, from eternity.
The wellspring of wisdom is the word of God in the heights,
and her runlets are the eternal commandments.

Bible - Joshua ben Sira

George Lemaître

- meaning to Friedmann's calculation
- unavoidable instability (with or without)
- experimental support in galaxy redshift

- A beginning for the universe: Genesis' *fiat lux* ?
  Pope Pius XII said the big bang was a god's manifestation.

- Lemaître reacted to this: he did not believe that any scientific true should be searched in the Bible, as scientific opinions in the Bible reflect only the common knowledge at the time of writing. He was aware of the precarious condition of every scientific truth: today's infinities may become finite tomorrow...

- Physics is about a quantitative description of the world.

- Infinities represent an end point for physics:
  their appearance means that we can not associate
  a finite number to the system under study.

- Infinities provide the most tantalizing paradoxes to physicists:
  they point to the old assumptions that we have to give away in
  order to go beyond our present theories, toward the explorations
  of new level of energy, space and time.

- Every infinity and its overcoming is the turning point
  for a new paradigm.

``I think that what is truly infinite may just be the abyss of our ignorance."

CARLO ROVELLI

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CARLO ROVELLI

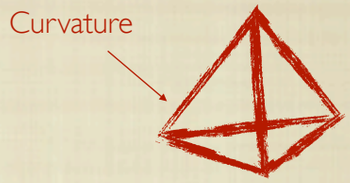
``Two things are infinite: the universe and human stupidity;
and I'm not sure about the universe."

ALBERT EINSTEIN

HISTORY OF THE MAIN IDEAS

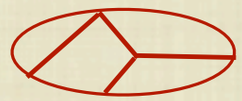
old quantum gravity

- 1957 $Z(q) = \int_{\partial g = q} Dg \;\; e^{iS_{EH}[g]}$ [Misner]
- 1961 Regge calculus → truncation of GR [Regge]
- 1967 W-DeW equation [Wheeler, DeWitt]
- 1971 Spin-geometry theorem → spin network [Penrose]



old LQG

- 1988 Complex variables for GR [Ashtekar]
- 1988 Loop solutions to WdW eq → LQG [Rovelli-Smolin]
- 1994 Spectral problem for geometrical operators → spin network
- 1996 Covariant dynamics → spinfoams [Reisenberger-Rovelli]



- 1999 LQC [Bojowald]

new results

- 2008 Covariant dynamics of LQG [Engle-Pereira-Livine-Rovelli, Freidel-Krasnov]
- 2010 Asymptotic of the new dynamics → recovery of Regge action [Conrady-Freidel, Barrett et al, Bianchi]

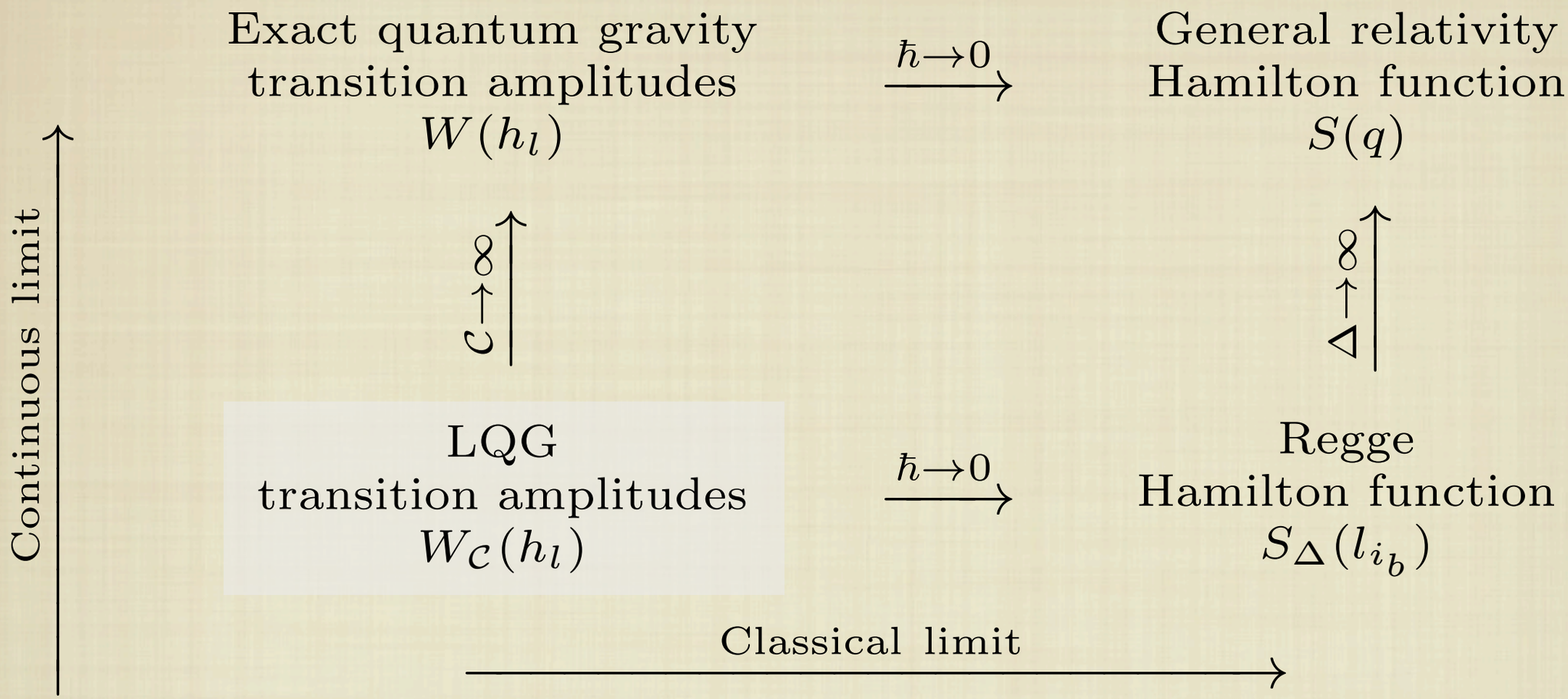
a "spinfoam"

- 2011 Cosmological constant → finiteness of the transition amplitudes [Han, Fairbairn-Moesburger]

→

- 2010 Spinfoam Cosmology [Bianchi-Rovelli-FV]

STRUCTURE OF THE THEORY



- No critical point
- No infinite renormalization
- Physical scale: Planck length

Regime of validity of the expansion:

$$L_{Planck} \ll L \ll \sqrt{\frac{1}{R}}$$